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SINGLE-PHASE PROBLEMS OF THE MELTING OF SOLID WEDGES

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Accurate solutions obtained in quasisteady formulation take the form of finite sums and are valid from a plane two-face aperture angle of kπ, where k is any simple fraction.

The present work is a continuation of [1] and uses the same notation and formulation of the problem of the melting of solids.

1. In Cartesian coordinates (y, z), the quasisteady heat-conduction equation takes the form [1]

$$\frac{\partial^2 U}{\partial y^2}+\frac{\partial^2 U}{\partial z^2}+\frac{V_0}{a}\,\frac{\partial U}{\partial z}=0. \tag{1}$$

The boundary conditions of the problem are

$$U(y,\ z)|_\Sigma=0,\quad U(y,\ z)\leqslant 0,\quad r_m\in\Omega, \tag{2}$$

$$U(y,\ z)\to U_\infty\leqslant 0\quad \text{as}\quad r_m\to\infty, \tag{3}$$

where Ω is the region of melted solid wedge. Dimensionless variables are introduced

$$\xi_j=(z\sin\Theta_j-y\cos\Theta_j)\,V_0/a,\quad j=1,\ 2. \tag{4}$$

The angles Θ_1 and Θ_2 are measured counterclockwise from the positive direction of the axis to a straight line passing through the corresponding face of the plane wedge (Fig. 1). The equations of the faces of the melting plane wedge here are: $\xi_1 = 0$, $\xi_2 = 0$; for the region inside the wedge, $\xi_1 > 0$, $\xi_2 > 0$. The straight line ξ_1 = const and ξ_2 = const are parallel to the corresponding planes of the wedge. Note that in geometric terms ξ_1 and ξ_2 are the distances from the point with coordinates (y, z) to the corresponding face of the wedge, multiplied by V_0/a. With this definition of Θ_1 and Θ_2, the aperture angle of the wedge φ_0 corresponds to the expression

$$\Theta_2-\Theta_1=\psi_0=\pi-\varphi_0. \tag{5}$$

The heat-conduction equation (1) is now written in new variables

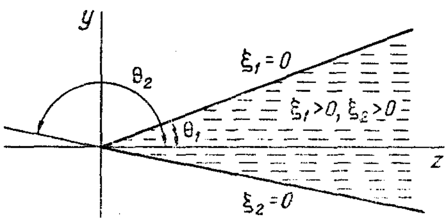

Fig. 1. Cross section of the melting wedge (shaded region).

Vinnitskii Polytechnic Institute. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 52, No. 6, pp. 995-999, June, 1987. Original article submitted January 2, 1986.

$$\frac{\partial^2 U}{\partial \xi_1^2}+\frac{\partial^2 U}{\partial \xi_2^2}+2B\frac{\partial^2 U}{\partial \xi_1 \partial \xi_2}+A_1\frac{\partial U}{\partial \xi_1}+A_2\frac{\partial U}{\partial \xi_2}=0, \tag{6}$$

$$A_1=\sin\Theta_1,\quad A_2=\sin\Theta_2,\quad B=\cos\psi,\quad \psi=\Theta_2-\Theta_1. \tag{7}$$

The fundamental solution of Eq. (6) is sought in the form

$$U=\exp-(\alpha\xi_1+\beta\xi_2). \tag{8}$$

Substituting Eq. (8) into Eq. (6), the characteristic equation obtained is

$$E(\alpha,\beta)=\alpha^2+\beta^2+2B\alpha\beta-A_1\alpha-A_2\beta=0. \tag{9}$$

The function of Eq. (8) is a fundamental solution of Eq. (6) in the case where the parameter α and β satisfy Eq. (9), i.e., the point (α, β) lies on the characteristic ellipse $E(\alpha, \beta) = 0$.

The algorithm for calculating the spectra $\{\alpha_n\}$ and $\{\beta_n\}$ ($n = 0, 1, 2, \ldots$) consists in the following procedure. Let $\alpha_0 = \beta_0 = 0$. The values α_1 and β_1 are found from the condition that the points $(\alpha_1\beta_1)$ and $(\alpha_1\beta_0)$ lie on the characteristic ellipse. From the equations $E(\alpha_0, \beta_1) = 0$, $E(\alpha_1, \beta_0) = 0$, it is found that $\alpha_1 = A_1$, $\beta_1 = A_2$. All the subsequent α_n and β_n when $n = 2, 3, \ldots$ are found from the equations

$$E(\alpha_{n\pm1},\beta_n)=0,\qquad E(\alpha_n,\beta_{n\pm1})=0. \tag{10}$$

From an analysis of the first pair of quadratic relations in Eq. (10), a recurrence relation is obtained for α_n, and from an analysis of the second pair of recurrence relation for β_n is obtained, in the form

$$\alpha_n=A_1-\alpha_{n-2}-2B\beta_{n-1},\quad \beta_n=A_2-\beta_{n-2}-2B\alpha_{n-1},\ n=2,\ 3,\ \ldots \tag{11}$$

Using the recurrence relations in Eq. (11), it may be proven that the coefficients α_n and β_n may be written in the form of linear and homogeneous dependences on A_1 and A_2

$$\alpha_n=A_1P_n(B)-A_2Q_n(B),\quad \beta_n=A_2P_n(B)-A_1Q_n(B), \tag{12}$$

where $P_n(B)$ is a polynomial of order $(n - 1)$ if n is odd and of order $(n - 2)$ if n is even; $Q_n(B)$ is a polynomial of order $(n - 1)$ if n is even and of order $(n - 2)$ of n is odd.

Substituting Eq. (12) into Eq. (11) gives the following recurrence formulas for P_n and Q_n

$$P_n=1-P_{n-2}+2BQ_{n-1},\quad Q_n=2BP_{n-1}-Q_{n-2},\ n=2,\ 3,\ \ldots \tag{13}$$

The expressions for the first seven polynomials P_n and Q_n are as follows

$$\begin{gathered}P_0=0,\quad P_1=P_2=1,\quad P_3=P_4=4B^2,\quad P_5=P_6=(1-4B^2)^2,\\ Q_0=Q_1=0,\quad Q_2=Q_3=2B,\quad Q_4=Q_5=2B(4B^2-1),\\ Q_6=Q_7=4B(1-4B^2)(1-2B^2).\end{gathered} \tag{14}$$

Using Eq. (13), the following properties may be proven for the polynomials P_n and Q_n by the induction method

$$P_{2m-1}=P_{2m},\quad Q_{2m-2}=Q_{2m-1},\quad m=1,\ 2,\ \ldots \tag{15}$$

For further investigation of the properties of the characteristic equation, the following quantities must be considered

$$\begin{gathered}S_n=2B(P_nP_{n-1}+Q_nQ_{n-1})-2P_nQ_{n-1}-2P_{n-1}Q_n+Q_n+Q_{n-1},\\ T_n=P_n^2+Q_n^2-2BP_nQ_n-P_n.\end{gathered} \tag{16}$$

It may be shown by induction that, for the polynomials P_n and Q_n from Eq. (13), S_n and T_n are zero for any n

$$S_n=0,\quad T_n=0,\quad n=1,\ 2,\ \ldots \tag{17}$$

The proof involves the definitions of P_n and Q_n from Eq. (13) and also the inductive assumption that $S_{n-1} = 0$, $T_{n-1} = 0$, and $T_{n-2} = 0$.

By definition, Eq. (8) is the fundamental solution when $\alpha = \alpha_n$ and $\beta = \beta_{n\pm1}$ or $\alpha = \alpha_{n\pm1}$ and $\beta = \beta_n$, when $n = 1, 2, \ldots$. In addition to such solutions, closing solutions are required, satisfying the conditions

$$\alpha = \alpha_n, \quad \beta = \beta_n, \quad E(\alpha_n, \beta_n) = 0. \tag{18}$$

Below, Eq. (18) is called the closure condition. The closure condition obtained from Eqs. (9) and (18) is

$$E(\alpha_n, \beta_n) = \alpha_n^2 + \beta_n^2 + 2B\alpha_n\beta_n - A_1\alpha_n - A_2\beta_n = 0. \tag{19}$$

The closure conditions in Eq. (19) imposes a constraint only on the difference of the angles $(\Theta_2 - \Theta_1)$. To prove this, expressions for α_n and β_n from Eq. (12) are substituted into Eq. (19). After transformations, it is found that

$$E(\alpha_n, \beta_n) = (A_1^2 + A_2^2)T_n - 2A_1A_2[2P_nQ_n - B(P_n^2 + Q_n^2) - Q_n] = 0. \tag{20}$$

Since $T_n = 0$, Eq. (20) simplifies to give

$$A_1A_2[2P_nQ_n - B(P_n^2 + Q_n^2) - Q_n] = 0. \tag{21}$$

Let $A_1A_2 \neq 0$ (the case when $A_1 = 0$ or $A_2 = 0$ is considered below). Then it follows from Eq. (21) that the closure condition in Eq. (18) does not depend on A_1 and A_2 and impose a constraint only on B, i.e., on the difference of angles $(\Theta_2 - \Theta_1)$. First setting $A_1 = A_2$ in Eq. (9), i.e., $\alpha_n = \beta_n$, and then $A_1 = -A_2$, i.e., $\alpha_n = -\beta_n$, the closure condition is simplified and breaks down into the following polynomial equations

$$P_n = Q_n \text{ and } (P_n - Q_n)(1 + B) = 1, \tag{22}$$

$$P_n = -Q_n \text{ and } (P_n + Q_n)(1 - B) = 1, \; n = 1, 2, \ldots \tag{23}$$

Two of the relations in Eqs. (22) and (23) must be regarded as the equations for finding the roots for B. The first relation in Eq. (22) or Eq. (23) is a polynomial of (n − 1)-th order and the second is a polynomial of n-th order. Thus, (2n − 1) values of B, denoted by B_m, and the corresponding angles ψ_m (m = 1, 2, ..., 2n − 1) may be determined from Eq. (22) or Eq. (23). It may be proven by induction that, for any n, one of the roots B_m is zero and correspondingly $\psi = \pi/2$. Analysis of the properties of the set of roots B_m and the corresponding angles ψ_m permits their distribution over the following two groups for specified n

$$\begin{aligned} &\text{I)} \quad \psi_m = \frac{m\pi}{n+1} = \pi - \psi_{(n-m+1)}, \quad B_m = -B_{n-m+1}, \; m = 1, 2, \ldots, n, \\ &\text{II)} \quad \psi_{(n+1)-(2n-1)} = \frac{(k-n)\pi}{n}, \quad k = n+1, n+2, \ldots, (2n-1). \end{aligned} \tag{24}$$

It is evident from Eq. (24) that all the ψ_m and B_m from the second group at the given n are repeated in the first group with (n − 1). For this reason, all the elements of the second group ψ_k and B_k must be omitted and only the elements ψ_m and B_m of the first group need be considered.

2. Suppose that the melting plane wedge has an aperture angle φ_0 such that

$$\psi_0 = \pi - \varphi_0 = \frac{m\pi}{n+1}, \tag{25}$$

where n is any natural number and m may take one of the values (1, 2, ..., n), i.e., the angle φ_0 must be written in terms of the aperture angle ψ_0 in the form $\psi_0 = \lambda\pi$, where λ is a regular fraction. Then the denominator of this fraction is (n + 1) and its numerator is m. In fact, the aperture angle of the wedge may take any rational value.

In this case, the accurate solution of the problem in Eqs. (1)-(3) takes the form of a sum

$$U = U_\infty \left\{1 + \sum_{p=1}^{n} (-1)^p [\langle \alpha_p\beta_{p-1} \rangle + \langle \alpha_{p-1}\beta_p \rangle] - (-1)^n \langle \alpha_n\beta_n \rangle \right\}, \tag{26}$$

where $\langle \alpha_i\beta_j \rangle = \exp - (\alpha_i\xi_1 + \beta_j\xi_2)$, $B = \cos\psi_0$, $A_1A_2 \neq 0$.

For specified angles Θ_1 and Θ_2 which determine the orientation of the wedge faces with respect to the z axis and the aperture angle of the wedge, B, A_1, A_2, ψ_0, m, n are calculated. Then the spectra $\{\alpha_i\}$ and $\{\beta_i\}$ are calculated from the recurrence formula in Eq. (11), with i = 2, 3, ..., n. Substituting the results into Eq. (26), the specific form of the accurate solution is obtained analytically.

It may be shown that the function in Eq. (26) constructed in this way is the accurate solution of the bounary problem in Eqs. (1)-(3). In fact, as a consequence of the closure conditions, the points (α_p, β_{p-1}), (α_{p-1}, β_p) $(p = 1, 2, \ldots, n)$ and the point $(\alpha_n \beta_n)$ lie by construction on the characteristic ellipse in Eq. (9). Therefore, the function in Eq. (26) satisfies the head-conduction equation (1).

It may be established by direct verification that, when $\xi_1 = 0$ or $\xi_2 = 0$, the right-hand side of Eq. (26) identically vanishes, i.e., the boundary conditions at Σ from Eq. (2) are satisfied.

With increasing distance from the faces of the wedge, $\xi_1 \to \infty$ or $\xi_2 \to \infty$, respectively. These properties of the variables ξ_1 and ξ_2 ensure that the inequality in Eq. (2) and the conditions in Eq. (3) are satisfied. In addition, the solution in Eq. (26) has the property that, on moving away from one face parallel to the other face, this solution transforms to the well-known one-dimensional solution of [2], of the type in Eq. (8).

Consider the case when $A_1A_2 = 0$, which corresponds to coincidence of one fact of the wedge with the z axis. Suppose that $A_2 = 0$, so as to be specific; then the closure condition in Eq. (18) will be satisfied for any $(\alpha_n \beta_n)$. In this case, the coefficients α_n and β_n are calculated from the formula

$$\alpha_n = A_1 P_n, \;\; \beta_n = -A_1 Q_n. \tag{27}$$

Using Eq. (27) and the properties of the polynomials P_n and Q_n in Eq. (15), it may be proven that in this case the right-hand side of Eq. (26) is identically zero for any $|B| < 1$. Hence the solution of Eqs. (1)-(3) in the form in Eq. (26) is only valid when $A_1A_2 \neq 0$.

The case when the melting wedge is symmetric relative to the z axis and the temperature is a field is also symmetric is of definite interest

$$\Theta_2 = \pi - \Theta_1, \;\; U(y, z) = U(-y, z). \tag{28}$$

Then the solution in Eq. (26) for the melting of a symmetric plane wedge with an aperture angle $2\Theta_1$ simplifies to the form

$$U = U_\infty \left\{1 + \sum_{p=1}^{n} (-1)^p 2 \left[\exp - \frac{V_0}{a}(\alpha_p + \alpha_{p-1}) z \sin\Theta_1\right] \times \right.$$
$$\left. \times \operatorname{Ch}\left[\frac{V_0}{a}(\alpha_p - \alpha_{p-1}) y \cos\Theta_1\right] - (-1)^n \exp - 2\alpha_n \frac{V_0}{a} z \sin\Theta_1\right\}.$$

NOTATION

U, temperature; Σ, surface of the melting plane wedge; Ω, region inside wedge; (y, z), rectangular Cartesian coordinates; V_0, velocity of translational motion of surface of melting wedge relative to material points of this wedge; a, thermal diffusivity; U_∞, temperature at points of the solid infinitely far from Σ; r_m, shortest distance from points of the body to Σ; ξ_1, ξ_2, auxiliary variables; Θ_1, Θ_2, angles of inclination of the wedge surfaces to the z axis; V_n, velocity of motion of the surface Σ projected onto its normal; n, normal to surface Σ; ρ, density; A_1, A_2, B, α_n, β_n, auxiliary constants; φ_0, aperture angle of wedge; P_n, Q_n, polynomials; m, n, natural numbers.

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